APPROXIMATE METHOD OF COMPUTATION OF THE NONSTATIONARY TURBULENT BOUNDARY LAYER IN AN INCOMPRESSIBLE LIQUID

(PRIBLIZHENNYI METOD RASCHETA NESTATSIONARNOGO TURBULENTNOGO POGRANICHNOGO SLOIA V NESZHIMAEMOI ZHIDKOSTI)

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L. A. ROZIN
(Leningrad)

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For an approximate computation of the stationary turbulent boundary layer, the assumption is sometimes made that the laminar and turbulent boundary layers are analogous [1]. This approach to the investigation of the turbulent motion of a liquid in the boundary layer permits certain important results to be obtained relatively simply. It therefore seems convenient to extend that analogy to the nonstationary boundary layer, particularly as the approximate methods of computing the stationary [2] and nonstationary [3] laminar boundary layers have much in common.

An approximate method of computation is presented, describing the nonstationary turbulent boundary layer, based on the assumed analogy between laminar and turbulent nonstationary layers. Examples are adduced which permit a number of valuable conclusions to be drawn.

1. Statement of the problem. We consider the mean velocity of the liquid in a turbulent boundary layer with the velocity of the external current a function of time. We shall assume that the structure of the turbulent pulsations and the character of the mean motion of the liquid at all times permit the assumption that the usual postulates defining the mean, used in the derivation of Reynolds' equation [2] are valid. That is, for the mean flow of the liquid in a turbulent boundary layer, the momentum equation can be written in the following form:

$$\frac{\partial \delta^{**}}{\partial x} + \frac{1}{U}\frac{\partial \delta^*}{\partial t} + \frac{1}{U}\frac{\partial U}{\partial x}(2\delta^{**} + \delta^*) + \frac{1}{U^2}\frac{\partial U}{\partial t}\delta^* = \frac{\tau_0}{\rho U^2} \tag{1.1}$$

Here, as usual,

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy, \qquad \delta^{**} = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \tag{1.2}$$

where $u(x, y, t)$ is the longitudinal velocity in the layer, $U(x, t)$ is the given velocity on the outer boundary of the layer, τ is the friction shear stress on the surface of the body in the fluid, ρ is the density of the fluid, x and y are the longitudinal and vertical coordinates respectively, and t is the time; the infinite limits in the integrals (1.2) correspond to the concept of an asymptotic boundary layer.

Following the procedures of the computation of the stationary turbulent layer [1], we multiply both sides of equation (1.1) by a certain function $G(R^*)$ of the Reynolds number R^* and introduce the symbols

$$\zeta = \frac{\tau_0}{\rho U^2} G(R^*), \qquad f = \frac{1}{U}\left(U \frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)\varphi, \qquad \varphi = \frac{\delta^* G(R^*)}{U} \qquad \left(R^* = \frac{U\delta^*}{\nu}\right) \tag{1.3}$$

where ν is the kinematic viscosity; equation (1.1) takes the form

$$G(R^*)\frac{\partial \delta^{**}}{\partial x} + \frac{1}{U} G(R^*) \frac{\partial \delta^*}{\partial t} + \left[\frac{\partial U}{\partial x}\left(2\frac{\delta^{**}}{\delta^*} + 1\right) + \frac{1}{U}\frac{\partial U}{\partial t}\right]\varphi = \zeta \tag{1.4}$$

Assuming $\delta^{**}/\delta^* = h$, the two first terms on the left-hand side of equation (1.4) can be expressed in the following form:

$$\frac{1}{U} G(R^*) \frac{\partial \delta^*}{\partial t} = \frac{\partial}{\partial t}\left[\frac{G(R^*)}{U}\delta^*\right] - \delta^* \frac{\partial}{\partial t}\left[\frac{G(R^*)}{U}\right] \tag{1.5}$$

$$G(R^*)\frac{\partial h\delta^*}{\partial x} = U\frac{\partial}{\partial x}\left[\frac{G(R^*)}{U}h\delta^*\right] - h\delta^* U \frac{\partial}{\partial x}\left[\frac{G(R^*)}{U}\right] \tag{1.6}$$

Introducing the notation

$$m(R^*) = \frac{G'(R^*)R^*}{G(R^*)} = \frac{d \log G(R^*)}{d \log R^*} \tag{1.7}$$

we write equations (1.5) and (1.6) as:

$$\frac{1}{U} G(R^*) \frac{\partial \delta^*}{\partial t} = \frac{1}{1+m}\left[\frac{\partial \varphi}{\partial t} + \frac{1}{U}\frac{\partial U}{\partial t}(1-m)\varphi\right] \tag{1.8}$$

$$G(R^*)\frac{\partial h\delta^*}{\partial x} = \frac{U}{1+m}\left[\frac{\partial h\varphi}{\partial x} + \frac{1}{U}\frac{\partial U}{\partial x}(1-m)h\varphi + m\frac{\partial h}{\partial x}\varphi\right] \tag{1.9}$$

Finally, substituting (1.8) and (1.9) into equation (1.4), we obtain

$$\frac{\partial \varphi}{\partial t} + U\frac{\partial h\varphi}{\partial x} + \left[\frac{\partial U}{\partial x}(3h + hm + m + 1) + 2\frac{1}{U}\frac{\partial U}{\partial t} + mU\frac{\partial h}{\partial x}\right]\varphi = (1+m)\zeta \tag{1.10}$$

We now consider the quantities f and ζ, defined by formulas (1.3). In the case of the laminar nonstationary boundary layer (3) the function $G(R^*)$ was put equal to R^*, and the quantity f was itself the parameter characterizing the form of the velocity profile in different cross-sections of the layer. There, the quantities ζ and h were functions of that parameter. We assume that in the case of the turbulent layer too the function $G(R^*)$ may be fixed in such a manner as to permit the use of f as the parameter of the velocity profile; and ζ and h are then considered as functions of f. Further, noticing that in the laminar boundary layer the quantity $G(R^*) = R^*$ is inversely proportional to the local coefficient of friction on the plate [2,3], in the turbulent case too we put $G(R^*) = (\rho U^2/\tau_0)_{f=0}$, where the value of R^* is taken for the body around which flow occurs. In making this assumption, we determine the form of the function $G(R^*)$ with the aid of the familiar power law for velocities and drag, which here takes the form

$$G(R^*) = 144.94\, R^{*^{1/4}} \tag{1.11}$$

Thus, taking $m(R^*) = 1/6$, and fixing $G(R^*)$ according to formula (1.11), we will presume that ζ and h are functions of f or ϕ. If we should succeed in establishing the form of these functions, equation (1.10) will be the differential equation in ϕ, the solution of which, by means of (1.3), will make it possible to determine R^* and therefore all the other basic quantities which characterize the boundary layer.

2. Choice of the functions $\zeta(f)$ and $h(f)$. Approximate computation of the nonstationary turbulent layer. As in the case of the stationary turbulent layer [1], in determining the functions $\zeta(f)$, and $h(f)$ we take advantage of the assumption of the analogous nature of the laminar and turbulent nonstationary layers. For this purpose, we normalize the parameter f in the laminar [3] and in the corresponding turbulent nonstationary layer in such a fashion that at the separation point its value is unity. Then

$$f^\circ = \frac{f}{f_s} \tag{2.1}$$

where f_s is the value of the non-normalized parameter f at the separation point, and $^\circ$ on f signifies the normalized quantity. We further normalize the functions ζ and h in such a manner that where $f = f^\circ = 0$ they become unity; that is

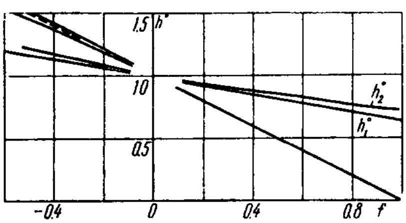
$$\zeta^\circ = \frac{\zeta}{(\zeta)_{f=0}}, \qquad h^\circ = \frac{h}{(h)_{f=0}} \tag{2.2}$$

In carrying out the above analogy, we assume that the functions $\zeta^\circ(f^\circ)$ and $h^\circ(f^\circ)$ have similar forms both in laminar and turbulent nonstationary boundary layers.

In computing the nonstationary laminar boundary layer [3], we have assumed two different types of functions ζ and h corresponding to two families of profiles of velocity in cross-section of the layer. In one of the velocity profiles, the usual Hartree functions have been used; in the other, the first approximation to the exact solution of the problem of the growth of the boundary layer having a free stream velocity of the form $U = t^n w(x)$. Now, the latter family of velocities, being taken from the nonstationary problem, leads to a more precise result for the time at which separation starts for flow past a circular cylinder. However, if we pass over to the normalized functions $\zeta^0(f^0)$ and $h^0(f^0)$, the difference between ζ_1^0, h_1^0 obtained by means of the Hartree functions, and ζ_2^0, h_2^0 taken from the solution of the indicated nonstationary problem, is negligible (Fig. 1). From the figure we see, first, that the functions h_1^0 and h_2^0 change very little, and may be taken as $h^0 = h_1^0 = 0.922$ (mean value). In cases where the separation of the boundary layer is determined, more precise results apparently give h^0 equal to the separation value of $h_1^0 = 0.642$. Passing to the function $\zeta^0(f^0)$, we put it in the form

$$\zeta^0(f^0) = 1 - f^0 + \varepsilon(f^0) \tag{2.3}$$

The figure shows the dotted straight line (2.3) for the case $\epsilon = 0$. The line falls between the continuous curves for ζ_1 (above) and ζ_2 (below), so that in the required interval of the values of f^0 the absolute magnitude of $\epsilon(f^0)$ is small, and in the great majority of cases we may assume that $\zeta^0(f^0)$ is a linear function.



Having established the form of the normalized functions ζ^0 and h^0 for the determination of ζ and h in the turbulent layer, it remains to develop the function f_s, the parameter of the turbulent layer at the point of separation, and the magnitude of h for $f = 0$. The latter quantity is taken as equal to $(h)_{f=0} = 0.714$ in most papers [2]. As far as f_s is concerned, it has not as yet been definitely established, and independent experimental determinations are needed to evalue it. Nevertheless, on the basis of data at hand [1], it is possible to write the following equation at the separation point:

$$\frac{1}{U^2}\left(U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)\delta^{**}\left(5.75\log\frac{U\delta^{**}}{\nu} + 3.8\right)^2 = -2 \tag{2.4}$$

which leads to the result $f_s = -5$, with the aid of (1.3) and $h = h^0(h)_{f=0} = 0.46$, where $h^0 = 0.642$ is taken at the separation point.

It now remains to write the expression for the functions h and ζ, which appear in equation (1.10). By use of the mean value of h^0 derived above, on the basis of the second formula (2.2), we have $h(f) = 0.66$. On the other hand, putting the assumption $\epsilon = 0$ into (2.3), and in the turbulent layer assuming $(\zeta)_{f=0} = 1$, from the first formula (2.2), we obtain

$$\zeta(f) = 1 - \frac{1}{f_s U}\left(U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)\varphi \tag{2.5}$$

Substituting the assumed h and ζ into equation (1.10), we finally write it in the form

$$\frac{\partial \varphi}{\partial t} + aU\frac{\partial \varphi}{\partial x} + \left(b\frac{\partial U}{\partial x} + c\frac{1}{U}\frac{\partial U}{\partial t}\right)\varphi = p \tag{2.6}$$

where

$$a = h = 0.66, \qquad b = 3h + hm + m + 1 + \frac{1}{f_s}(1+m) = 3.02 \tag{2.7}$$

$$c = 2 + \frac{1}{f_s}(1+m) = 1.77 \qquad p = 1 + m = 1.17$$

Differential equation (2.6) is analogous to the corresponding equation assumed as the basis of the computation of the laminar nonstationary boundary layer [3]. This fact is the consequence of the assumption as to the analogous nature of laminar and turbulent nonstationary layers.

Intergration of equation (2.6) under suitable conditions solves the problem of the approximate computation of the nonstationary turbulent boundary layer. In practice, if ϕ is known, it can be used to determine R^*, with the aid of (1.3) and (1.11), according to the formula

$$R^* = \left(0.006899\,\frac{\varphi U^2}{\nu}\right)^{4/7} \tag{2.8}$$

Thereupon the quantity δ^* is derived from R^*, and in turn, $\delta^{**} = h\delta^*$. The shear stress on the wetted surface τ_0 is expressed in the following manner:

$$\tau_0 = \rho U^2\frac{\zeta(f)}{G(R^*)} = \mu\varphi^{-1/7}\left(0.006899\,\frac{U^2}{\nu}\right)^{6/7}\left[1 - \frac{1}{f_s U}\left(U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)\varphi\right] \tag{2.9}$$

From this, the condition of separation of the boundary layer $\tau_0 = 0$, is written in the form

$$\frac{1}{U}\left(U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right)\varphi = f_s \tag{2.10}$$

All the foregoing considerations are readily generalized for the case of the axisymmetrical nonstationary turbulent boundary layer. Indeed, again using the momentum equation, which would differ from (1.1) only by the presence on the left-hand side of the additional term $(1/r)(dr/dx)\delta^{**}$, we thus obtain the following differential equation for ϕ:

$$\frac{\partial\varphi}{\partial t} + aU\frac{\partial\varphi}{\partial x} + \left(b\frac{\partial U}{\partial x} + c\frac{1}{U}\frac{\partial U}{\partial t} + q\frac{1}{r}\frac{dr}{dx}U\right)\varphi = p \tag{2.11}$$

where $q = h(1 + m) = 0.771$ and r is the radius of any of the parallel contours of the wetted surface.

3. Examples. We pass to a consideration of several examples.

1. Owing to the fact that equation (2.6) and the corresponding equation of laminar motion [3] have the same form, classes of problems for which (2.6) becomes an ordinary differential equation will be the same as in reference [3]. Without writing out all the equations that pertain to this problem, we dwell on the particular example of liquid motion at the forward part of a blunt obstacle where the velocity at the edge of the boundary layer is expressed by the formula

$$U = U_0 k(\xi)\, x\lambda, \qquad \xi = tU_0\lambda \tag{3.1}$$

where the constants U_0 and λ have the dimensions $[U_0] = LT^{-1}$, $[\lambda] = L^{-1}$, and L and T are the dimensions of length and time.

In this case it is easy to obtain the following expression for ϕ:

$$\varphi = \frac{1}{\lambda U_0 k^c}\exp\left(-b\int k\, d\xi\right)\left[D + p\int_0^\xi k^c \exp\left(b\int k\, d\xi\right)d\xi\right] \tag{3.2}$$

where D is a constant of integration. If the motion of the body in the liquid starts from rest, from the condition that ϕ be finite for $t = 0$ and $k = 0$ it follows that $D = 0$. For such a motion, the condition of separation of the boundary layer is written in the form

$$\frac{p}{k^{1+c}}\left(k^2 + \frac{dk}{d\xi}\right)\exp\left(-b\int k\, d\xi\right)\int_0^\xi k^c \exp\left(b\int k\, d\xi\right)d\xi = f_s \tag{3.3}$$

correct both for laminar and turbulent flow. Equation (3.3) permits the conclusion that separation of the boundary layer in the case considered takes place at all x simultaneously. Hence, by comparing the coefficients p, b, c and f_s in the turbulent and corresponding laminar layers (for the Hartree profile $p = 1.12$, $b = 2.422$, $c = 1.00$, $f_s = -1.12$), we may conclude that for similar distributions of pressure the turbulent layer will separate considerably later than the laminar layer, since in the former case $|f_s|$ is almost 4.5 times as large as in the latter.

If at a certain moment of time $t = \xi = 0$ the boundary layer was laminar, the constant D in (3.2) will be determined by the following equation:

$$D = D_1 = \frac{\nu}{U_0 \lambda x^2}\left[k^{c-2}\exp\left(b\int k\,d\xi\right)\right]_0 R_0{}^* G(R_0{}^*) \qquad (3.4)$$

Here the index 0 signifies that the quantity in question is chosen at $t = \xi = 0$, and by $R_0{}^*$ is meant the value of Reynolds' number $U\delta^*/\nu$ at instant $t = 0$, computed according to the theory of the laminar boundary layer. On the other hand, for the same distribution of pressure, but for the conditions of turbulent motion only, the quantity $D = D_2$ is also expressed according to formula (3.4), in which, however, $R_0{}^*$ is computed according to the theory of the turbulent boundary layer. Since $R_0{}^*$ in the laminar layer is considerably less than $R_0{}^*$ of the turbulent layer, the following inequality will be true: $D_1 < D_2$. On the basis of formulas (2.10) and (3.2), this inequality leads to the conclusion that in the presence of laminar motion up to some instant of time separation occurs later than in its absence.

2. We assume that at the initial instant of time, $t = 0$, a semi-infinite plate begins to move with relation to the ambient liquid with a free stream speed $U = U_0 + \sigma(1 - \cos\omega t)$. We will assume that on the background of velocity U_0 small oscillations are superposed ($U_0 \gg \sigma$), the frequency of which, ω, is considerably less than the frequency of the turbulent pulsations. This type of problem has been considered for the laminar boundary layer [3], and for the parameter f the following expressions have been obtained:

$$f = \frac{p}{a}\frac{\sigma\omega}{U_0{}^2}x\sin\omega t \quad (x < aU_0 t), \qquad f = p\frac{\sigma\omega}{U_0}t\sin\omega t \quad (x > aU_0 t) \qquad (3.5)$$

where the origin of x is set at the leading edge of the plate. It is obvious that in the case of the turbulent boundary layer, the same expression for f can be obtained with the coefficients a, p fixed by the formula (2.7). Because of the similar nature of the solution of the problem, this permits a verification of the difference between laminar and turbulent plate flow. From formula (3.5) it follows that two regions form on the plate, separated by a moving boundary $x = aU_0 t$. On one side of the boundary ($x < aU_0 t$), the boundary layer thickens, that is, the parameter f, and consequently δ^*, depends on x; on the other side of the line ($x > aU_0 t$) the influence of the leading edge of the plate no longer has any effect, and δ^* changes only with time. Little by little the developing boundary layer covers the whole plate; the velocity of motion of the border of the indicated regions is equal to aU_0 and is almost twice as great for turbulent as for laminar plate flowl

Along with the development of the boundary layer, separation also takes place. Actually it is possible to conclude by use of equation (3.5) that separation begins for $x > aU_0 t$ at the instant of time t_0 determined by the condition

$$t_0^\circ \sin N_{sh} t_0^\circ = -\beta \frac{U_0}{\sigma} \frac{1}{N_{sh}} \qquad \left(t_0^\circ = \frac{U_0}{L} t_0,\ N_{sh} = \frac{\omega L}{U_0}\right) \tag{3.6}$$

Here L is the scale of length, N_{sh} is the Strouhal number, and β is the coefficient equal to 1.0 and 4.27 for laminar and turbulent layers respectively. It also follows, other things being equal, that the nonstationary turbulent boundary layer separates later than the laminar layer.

Taking into account (3.5) and the expression (2.9), the shear stress and the drag F of a plate of length L washed by a current of liquid on both sides can be determined. If we designate the wetted surface of the plate by S, and introduce the coefficient of drag by using the formula $F = C_f S\ 1/2\ \rho\ U_0^2$, for C_f we obtain the following expression:

$$C_f = R^{-1/7} t^{\circ 6/7} \left[\left(0.00303 + \frac{0.0275}{t^\circ}\right) + \left(\frac{0.00644}{t^\circ} - 0.00195\right) \frac{\sigma}{U_0} N_{sh} t^\circ \sin N_{sh} t^\circ\right] \tag{3.7}$$

where $R = U_0 L/\nu$. At the instant $t^0 = 1.51$ the whole plate of length L falls into the region in which the flow depends on x, and from that instant the formula for C_f takes the form:

$$C_f = R^{-1/7} \left(0.0303 + 0.00496 \frac{\sigma}{U_0} N_{sh} \sin N_{sh} t^\circ\right) \tag{3.8}$$

In the case $\sigma = 0$, formula (3.7) corresponds to the development of a boundary layer on a plate which begins to move instantaneously from the state of rest, with a constant velocity U_0. For this case, at the instant $t^0 = 1.51$, when steady motion has been established over the entire plate, $C_f = 0.0303\ R^{-1/7}$, which agrees with the usual formula for drag [2]. Putting $\sigma = 0$, we evaluate the additional drag force due to the unsteady flow. For this purpose we fix the quantity A equal to the impulse of the drag force F for the time $t^0 = 1.51$. Then, considering unsteady motion, we have $A = 0.0497\,\mu\,LR^{6/7}$. On the other hand, for steady motion, we obtain $A = 0.0461\,\mu\,LR^{6/7}$. It is evident from this that the magnitude of the impulse of the force F in unsteady motion is 8 per cent greater than the corresponding magnitude for steady motion. For laminar motion [3] this excess impulse of the force F amounts to almost 33 per cent.

In conclusion two remarks must be made.

1. Recently Loitsianskii has surmised that near the point of separation $C = (R^*)$ const, since here $\tau_0 = 0$, and the influence of viscosity and thus of the Reynolds number R^* vanishes. This assumption has an insignificant effect on the coefficients of the equations (1.10) and (2.6), and consequently hardly influences f or ϕ. At the same time, however, the size of the displacement δ^* near the separation point is already

determined not by formula (2.8), but by another method described in monograph [2].

2. In the case of laminar flow [3], and likewise for turbulent flow, in the first approximation for simplification of the computation of the boundary layer, it has been assumed that $h = \delta^{**}/\delta^{*}$ = const. Turning to the integral relations of impulses (1.1), it is easy to conclude that this assumption is more justified when h is less important in equation (1.1). This in turn will be the situation when in strongly unsteady flow the magnitudes characterizing the boundary layer change with time faster than x, and $\partial U/\partial x$ is small in comparison with $(\partial U/\partial t)/U$. For more precise computations, both of laminar and of turbulent boundary layers, the assumption of a constant ratio δ^{**}/δ^{*} and the approximation that $h(f)$ is a linear function is not correct; in that case the differential equation involving ϕ is nonlinear.

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